

A sufficient condition for a discrete spectrum of the Kirchhoff plate with an infinite peak*

F.L. Bakharev, S.A. Nazarov, G.H. Sweers

Abstract

Sufficient conditions for a discrete spectrum of the biharmonic equation in a two-dimensional peak-shaped domain are established. Different boundary conditions from Kirchhoff's plate theory are imposed on the boundary and the results depend both on the type of boundary conditions and the sharpness exponent of the peak.

Keywords: Kirchhoff plate, cusp, peak, discrete and continuous spectra.

1 Motivation

Elliptic boundary value problems on domains which have a Lipschitz boundary and a compact closure, in particular when they generate positive self-adjoint operators, have fully discrete spectra. However, if the domain loses the Lipschitz property or compactness, other situations may occur. It is well-known that for the Dirichlet case boundedness is sufficient but not necessary for a discrete spectrum. See the famous paper by Rellich [26] or the more recent contributions by [27, 3]. On the other hand, for the Neumann problem of the Laplace operator there exist numerous examples of bounded domains such that the spectrum gets a non-empty continuous component (see e.g. [7, 17, 18, 28, 11]).

The literature on the spectra for the Laplace operator with various boundary conditions is focussed on domains that have a cusp, a finite or infinite peak or horn [3, 11, 13, 8, 14, 12, 6, 4, 15] or even considered a rolled horn [28].

The criteria in [1] and [9] for the embedding $H^1(\Omega) \subset L_2(\Omega)$ to be compact, show that the Neumann-Laplace problem on a domain Ω with the infinite peak

$$\Pi_R = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > R, -H(x_1) < x_2 < H(x_1)\}, \quad (1)$$

where the function $H > 0$ is smooth and monotone decreasing, has a discrete spectrum if and only if

$$\lim_{y \rightarrow +\infty} \int_y^{+\infty} \frac{H(\eta)}{H(y)} d\eta = 0 \quad (\Leftrightarrow \quad \lim_{y \rightarrow +\infty} \frac{H(y + \epsilon)}{H(y)} = 0 \text{ for any } \epsilon > 0). \quad (2)$$

The function H is assumed to have a first derivative that tends to zero and a bounded second derivative. It will be convenient to use the notation $\Upsilon(y) = (-H(y), H(y))$. Here is an image of such a domain:



*The work is supported by the Russian Foundation for Basic Research (project 12-01-00348). The first author is also supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, Saint-Petersburg State University) under the grant 11.G34.31.0026 of the Government of the Russian Federation and grant 6.38.64.2012 of Saint-Petersburg State University.

The simplest boundary irregularity violating the Lipschitz condition, is but the (finite) peak

$$\varpi_R = \{x : 0 < x_1 < R, -h(x_1) < x_2 < h(x_1)\}, \quad (3)$$

where $h(x_1) = h_0 x_1^{1+\alpha}$, $h_0 > 0$ and $\alpha > 0$. Nevertheless, the spectrum of the Neumann problem in the domain with this peak stays discrete. See Remark 1.1.

A criterion ([25]) for an essential spectrum in the Neumann problem for elliptic systems of second order differential equations with a polynomial property is derived in [21]. In particular it shows that the continuous spectrum of an elastic body with $\alpha \geq 1$ for the peak (3), is non-empty (see [23, 2]). This phenomenon of generating wave processes in a finite volume, is known experimentally and used in the engineering practice to construct wave dampers, “black holes”, for elastic oscillations (see [20, 16], etc.).

In this paper we study the spectra of boundary value problems for the Kirchhoff model of a thin elastic plate described by the biharmonic operator Δ^2 . We consider the three mechanically most reasonable cases, namely where the lateral sides of the peak are supplied with one of the following three types of the boundary conditions: the clamped edge (*Dirichlet*), the traction-free edge (*Neumann*) and the hinged edge (*Mixed*). In all the cases the spectrum of the problem in a bounded domain with the peak (3) is discrete. We derive sufficient conditions for the spectrum to be discrete for the boundary value problem on an unbounded domain with a peak as in (1).

If a “sufficient number” of Dirichlet conditions are imposed on the lateral sides of the peak (the cases *D–D*, *D–M*, *D–N* and *M–M*; see formulas (5–7) and (10), (11)), then the proof that the spectrum is discrete becomes rather simple (Theorem 1). Indeed, it suffices to apply the weighted Friedrich’s inequality (12) and to take into account the decay of the quantity $H(y)$ as $y \rightarrow +\infty$. With a different argument, it is straightforward to obtain a condition for the discrete spectrum if the peak’s edges are traction-free (the case *N–N*; see Corollary 9). Indeed, as shown in [1], the second condition in (2) implies a criterion for the compact embedding $H^m(\Omega) \subset L^2(\Omega)$ for all m (we just need $m = 2$). Therefore, we are only interested into investigating the case *M–N*, i.e., one side of the peak is traction-free and the other hinged. We obtain a sufficient condition for the discrete spectrum (Theorem 10) by applying weighted inequalities of Hardy type (Lemmas 3 and 4). This approach differs from the one used in [1, 9], and we can also use it for the case *N–N* (Theorem 8).

The obtained results essentially differ from each other: under the conditions (10) and also under (11) any decay of H is enough, but the case *M–N* needs a power decay rate with the exponent $\alpha > 1$ while the case *N–N* needs a superexponential decay rate (see comments to Theorems 8 and 10).

2 The Kirchhoff plate model

Let Ω be a domain in the plane \mathbb{R}^2 with a smooth (of class C^∞) boundary $\partial\Omega$, coinciding with the peak (1) inside the half-plane $\{x : x_1 > R\}$ and being bounded outside this half-plane. We regard Ω as the projection of a thin isotropic homogeneous plate and apply the Kirchhoff theory (see [19, §30], [22, Ch.7] etc). So we arrive at the fourth-order differential equation

$$\Delta^2 u(x) = \lambda u(x), \quad x \in \Omega, \quad (4)$$

which describes transverse oscillations of the plate. Here, $u(x)$ is the plate deflection, and λ a spectral parameter proportional to the square of the oscillation frequency.

The following sets of boundary conditions have a clear physical interpretation (see [19, §30], [10, §1.1] etc):

(D) Dirichlet for a clamped edge:

$$u(x) = \partial_n u(x) = 0, \quad x \in \Gamma_D. \quad (5)$$

(N) Neumann for a traction-free edge:

$$\begin{cases} \partial_n \Delta u(x) - (1 - \nu)(\partial_s \kappa(x) \partial_s u(x) - \partial_s^2 \partial_n u(x)) = 0, \\ \Delta u(x) - (1 - \nu)(\partial_s^2 u(x) + \kappa(x) \partial_n u(x)) = 0, \end{cases} \quad x \in \Gamma_N. \quad (6)$$

(M) Mixed for a hinged edge:

$$u = \Delta u(x) - (1 - \nu) \kappa(x) \partial_n u(x) = 0, \quad x \in \Gamma_M. \quad (7)$$

Here, ∂_n and ∂_s stand for the normal and tangential derivatives, $\kappa(x)$ is the signed curvature of the contour Γ at the point $x \in \Gamma$ positive for convex boundary parts, and $\nu \in [0, 1/2)$ is the Poisson ratio. Finally, Γ_D , Γ_N , and Γ_M are the unions of finite families of open curves and $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_M}$, two of which may be empty. In what follows it is convenient to use the notation $y = x_1$ and $z = x_2$.

The general properties of the spectra depend on which of the boundary conditions (5–7) are imposed on the upper (+) and lower (–) sides $\Sigma_R^\pm = \{x : y > R, z = \pm H(y)\}$ of the peak. To be more precise, we consider the spectral problem of the variational formulation

$$a(u, v) = \lambda(u, v)_\Omega, \quad v \in \mathbb{H}, \quad (8)$$

where $(\cdot, \cdot)_\Omega$ is a scalar product in the Lebesgue space $L_2(\Omega)$, a is the symmetric bilinear form such that $\frac{1}{2}a(u, u)$ is the elastic energy stored in the plate:

$$a(u, u) = \int_\Omega \left(\left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 u}{\partial x_2^2} \right|^2 + 2(1 - \nu) \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 + 2\nu \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 u}{\partial x_2^2} \right) dx, \quad (9)$$

and \mathbb{H} is the subspace of functions $u \in H^2(\Omega)$, satisfying the conditions (5) on Γ_D and $u = 0$ on Γ_M in the sense of traces. One directly verifies that

$$a(u, u) \geq (1 - \nu) \sum_{j,k=1}^2 \int_\Omega \left| \frac{\partial^2 u}{\partial x_j \partial x_k} \right|^2 dx.$$

Hence, the bilinear form on the left-hand side of (8) is positive and closed on $\mathbb{H} \times \mathbb{H}$, and according to [5, §10.1] we can associate a unbounded positive self-adjoint operator A in the Hilbert space $L_2(\Omega)$ to problem (8) with domain $D(A) = \mathbb{H}$. Its spectrum σ belongs to $[0, \infty)$ and, moreover, the spectrum is discrete if and only if the embedding $\mathbb{H} \subset L_2(\Omega)$ is compact (cf. [5, Theorem 10.1.5]). This is what we call the spectrum for (4–7) and which will be studied in the present paper. The remaining boundary conditions in (5–7) appear as intrinsic natural conditions from (8–9), see again [19, §30], [10, §1.1] etc.

3 Simple cases

Assume that the chosen boundary conditions provide at least one of the following two groups of relations

$$i) \quad u = 0 \quad \text{on } \Sigma_R^+ \cup \Sigma_R^-, \quad (10)$$

$$ii) \quad u = \partial_n u = 0 \quad \text{on } \Sigma_R^+ \quad (\text{or on } \Sigma_R^-). \quad (11)$$

In both the cases (10) and (11) the following version of Friedrich's inequality is valid

$$\int_{\Upsilon(y)} \left| \frac{\partial^2 u}{\partial z^2}(y, z) \right|^2 dz \geq \frac{c}{H(y)^4} \int_{\Upsilon(y)} |u(y, z)|^2 dz, \quad (12)$$

and, therefore,

$$a(u, u) \geq c \int_{\Omega} H(y)^{-4} |u(x)|^2 dx. \quad (13)$$

The embedding operator $\gamma : \mathbb{H} \rightarrow L^2(\Omega)$ can be represented as the sum $\gamma_0 + \gamma_\rho$, where $\rho \geq R$ is large positive, $\gamma_0 = \gamma - \gamma_\rho$, and γ_ρ includes the operator of multiplication by the characteristic function of Π_ρ . The operator γ_0 is compact, and the norm of γ_ρ , in view of (13), does not exceed $c \max \{H(x_1)^{-2}; x_1 \geq \rho\}$. Since the function H decays, this quantity goes to zero when $\rho \rightarrow +\infty$, i.e., the operator γ can be approximated by compact operators in the operator norm. Thus, γ is compact and the following result is proved.

Theorem 1 *Suppose that the boundary conditions for problem (4) as given in (5-7) contain one of the cases (10) or (11). Then the spectrum is discrete.*

Remark 1.1 *By a similar reasoning, we may conclude that in the bounded domain ω , with the peak as in (3), equation (4) has a discrete spectrum for any set of conditions (5-7) on the arc $\partial\omega \setminus \mathcal{O}$. This fact follows from the inequality (see [24]):*

$$\left\| |x|^{-1} u; L_2(\omega) \right\|^2 \leq c \left(\|\nabla u; L_2(\omega)\|^2 + \|u; L_2(\omega \setminus \varpi_R)\|^2 \right).$$

4 Auxiliary inequalities

First of all we prove some one-dimensional weighted inequalities, two of which are of Hardy type involving a weight function h as follows.

Assumption 2 *Let h be a positive weight function of class C^2 on $[0, +\infty)$ satisfying*

- $\int_0^\infty h(s) ds < \infty$ and
- for some T large $h'(t) < 0$ and $h''(t) > 0$ for $t \in (T, \infty)$.

Throughout this section h is supposed to satisfy this assumption.

Lemma 3 *If U is differentiable for $y \geq R$ and $U(R) = 0$, then*

$$\int_R^{+\infty} h(y) |U(y)|^2 dy \leq \int_R^{+\infty} F_h(y) |\partial_y U(y)|^2 dy,$$

where $F_h(y) = \frac{4}{h(y)} \left(\int_y^{+\infty} h(\tau) d\tau \right)^2$.

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\begin{aligned} \int_R^{+\infty} h(y) |U(y)|^2 dy &= 2 \int_R^{+\infty} h(y) \int_R^y \partial_t U(t) U(t) dt dy \leq \\ &\leq 2 \int_R^{+\infty} \int_t^{+\infty} h(y) |\partial_t U(t) U(t)| dy dt \leq \\ &2 \left(\int_R^{+\infty} h(t) |U(t)|^2 dt \right)^{1/2} \left(\int_R^{+\infty} h(t)^{-1} \left(\int_t^{+\infty} h(y) dy \right)^2 |\partial_t U(t)|^2 dt \right)^{1/2}, \end{aligned}$$

and the result follows through division by a common factor. \square

Lemma 4 *If U is differentiable for $y \geq R$ and $U(R) = 0$, then*

$$\int_R^{+\infty} h(y) |\partial_y U(y)|^2 dy \geq \int_R^{+\infty} G_h(y) |U(y)|^2 dy,$$

where $G_{h,R}(y) = \frac{1}{4h(y)} \left(\int_R^y h(\tau)^{-1} d\tau \right)^{-2}$.

Proof. For functions v with $v(0) = 0$ a Hardy inequality tells us that

$$\int_0^{+\infty} t^{-2} |v(t)|^2 dt \leq 4 \int_0^{+\infty} |\partial_t v(t)|^2 dt.$$

We make the change $t \mapsto y \in [R, +\infty)$ where $t = \int_R^y h(\tau)^{-1} d\tau$, and set $U(y) = v(t)$. Then $\partial_t v(t) = h(y) \partial_y U(y)$ leads to the desired estimate. \square

Corollary 5 *If the function U is twice differentiable for $y \geq R$ and $U(R) = U'(R) = 0$, then*

$$\int_R^{+\infty} h(t) |\partial_t^2 U(t)|^2 dt \geq W_h(R) \int_R^{+\infty} h(t) |U(t)|^2 dt \quad (14)$$

is valid with

$$W_h(R) := \inf_{t \in [R, +\infty)} \frac{G_{h,R}(t)}{F_h(t)} = \inf_{t \in [R, +\infty)} \frac{1}{16} \left(\int_R^t h(\tau)^{-1} d\tau \right)^{-2} \left(\int_t^\infty h(\tau) d\tau \right)^{-2}. \quad (15)$$

The following lemma will show that the assumptions in Theorem 8 can be met.

Lemma 6 *We have*

$$\inf_{t \in [R, +\infty)} \frac{G_{h^3,R}(t)}{h(t) h'(t)^2} \geq 1. \quad (16)$$

Suppose moreover that

$$\lim_{t \rightarrow +\infty} \partial_t (\log(h(t))) = -\infty. \quad (17)$$

Then

$$W_h(R) \rightarrow +\infty \text{ and } W_{h^3}(R) \rightarrow +\infty \text{ for } R \rightarrow \infty.$$

Proof. Since $-h'(\tau) \geq -h'(t) > 0$ for $\tau < t$, we find

$$\begin{aligned} \frac{G_{h^3,R}(t)}{h(t) h'(t)^2} &= \frac{1}{4h'(t)^2 h(t)^4} \left(\int_R^t h(\tau)^{-3} d\tau \right)^{-2} \geq \\ &\geq \frac{1}{4h(t)^4} \left(- \int_R^t h'(\tau) h(\tau)^{-3} d\tau \right)^{-2} = \frac{1}{4h(t)^4} \left(\frac{1}{2h(t)^2} - \frac{1}{2h(R)^2} \right)^{-2} \geq 1. \end{aligned}$$

Since (17) equals $h'(t)/h(t) \rightarrow -\infty$ for $t \rightarrow \infty$, we find that for $t \rightarrow \infty$ both

$$h(t) \left(\int_t^\infty h(\tau) d\tau \right)^{-1} \rightarrow \infty \text{ and } \frac{1}{h(t)} \left(\int_R^t \frac{1}{h(\tau)} d\tau \right)^{-1} \rightarrow \infty.$$

Hence $G_{h,R}(t)/F_h(t) \rightarrow \infty$ for $t \rightarrow \infty$ and since the quotient also goes to infinity for $t \downarrow R$, it has a minimum in some $t_R \in (R, \infty)$. Calculating $(G_{h,R}(t)/F_h(t))' = 0$ we find

$$\frac{1}{h(t)} \int_t^\infty h(\tau) d\tau - h(t) \int_R^t h(\tau)^{-1} d\tau = 0.$$

Hence

$$\inf_{t \in [R, \infty)} \left(\int_R^t h(\tau)^{-1} d\tau \int_t^\infty h(\tau) d\tau \right)^{-1} = h(t_R)^2 \left(\int_{t_R}^\infty h(\tau) d\tau \right)^{-2},$$

which goes to infinity for $R \rightarrow \infty$ since $t_R > R$. The claim for $W_h(R)$ follows. The same derivation holds true for $W_{h^3}(R)$. \square

5 The traction-free boundary and a 2d-estimate

We assume that the Neumann boundary conditions (6) are imposed at the both sides of the peak (1). While $\rho \rightarrow +\infty$ let us describe the behavior of multiplier $K(\rho)$ in the inequality

$$K(\rho) \int_{\Pi_\rho} |u(y, z)|^2 dy \leq \|u; H^2(\Omega)\|^2, \quad u \in H^2(\Omega). \quad (18)$$

If $K(\rho)$ increases unboundedly as $\rho \rightarrow +\infty$ then, as above, Theorem 10.1.5 [5] ensures that the spectrum of the problem (4–7) stays discrete even in the case both sides of the peak are supplied with the traction-free boundary conditions (N) and, moreover, for any other boundary conditions from the list (5–7).

Proposition 7 *Suppose that Assumption 2 is satisfied and that*

$$\lim_{t \rightarrow \infty} \partial_t (\log (H(t))) = -\infty.$$

Then for ρ sufficiently large (18) holds true with

$$K(\rho) = c \min\{H^{-4}(\rho), W_H(\rho), W_{H^3}(\rho)\}.$$

Proof. It is sufficient to check the inequality (18) for smooth functions which vanish for $y < \rho$. We use the representation

$$u(x) = u(y, z) = u_0(y) + zu_1(y) + u^\perp(y, z)$$

where, for $y > \rho$, the component u^\perp is subject to the following two orthogonality conditions

$$\int_{\Upsilon(y)} u^\perp(y, z) dz = 0 \text{ and } \int_{\Upsilon(y)} \partial_z u^\perp(y, z) dz = u^\perp(y, H(y)) - u^\perp(y, -H(y)) = 0. \quad (19)$$

Let us process the integrals on the right-hand side of

$$\int_{\Pi_\rho} |\nabla_x^2 u(x)|^2 dx = I_1 + 4I_2 + I_3, \text{ where} \quad (20)$$

$$I_1 := \int_{\Pi_\rho} |\partial_z^2 u(x)|^2 dx, \quad I_2 := \int_{\Pi_\rho} |\partial_y \partial_z u(x)|^2 dx, \quad I_3 := \int_{\Pi_\rho} |\partial_y^2 u(x)|^2 dx.$$

Since $I_1 = \int_\rho^{+\infty} \int_{\Upsilon(y)} |\partial_z^2 u^\perp(y, z)|^2 dz dy$, since by the orthogonality conditions in (19) also here inequality (12) holds, we find

$$I_1 \geq c \int_{\Pi_\rho} H(y)^{-4} |u^\perp(x)|^2 dx. \quad (21)$$

For the last term in (20) we have

$$I_3 = \int_{\Pi_\rho} \left| \partial_y^2 u_0(y) + z \partial_y^2 u_1(y) + \partial_y^2 u^\perp(y, z) \right|^2 dx \geq J_1 + J_2 + 2J_3 + 2J_4, \quad (22)$$

where

$$J_1 = \int_{\Pi_\rho} |\partial_y^2 u_0(y)|^2 dx, \quad J_3 = \int_{\Pi_\rho} \partial_y^2 u_0(y) \partial_y^2 u^\perp(y, z) dx,$$

$$J_2 = \int_{\Pi_\rho} |z \partial_y^2 u_1(y)|^2 dx, \quad J_4 = \int_{\Pi_\rho} z \partial_y^2 u_1(y) \partial_y^2 u^\perp(y, z) dx.$$

We readily notice that according to the inequality (14) the estimates

$$J_1 \geq W_H(\rho) \int_\rho^{+\infty} 2H(y) |u_0(y)|^2 dy = W_H(\rho) \int_{\Pi_\rho} |u_0(y)|^2 dx, \quad (23)$$

$$J_2 = \frac{2}{3} \int_\rho^{+\infty} H^3(y) |\partial_y^2 u_1(y)|^2 dy \geq$$

$$\geq \frac{2}{3} W_{H^3}(\rho) \int_\rho^{+\infty} H^3(y) |u_1(y)|^2 dy = W_{H^3}(\rho) \int_{\Pi_\rho} |zu_1(y)|^2 dx \quad (24)$$

are fulfilled. For our purpose we need $W_H(\rho) \rightarrow +\infty$ and $W_{H^3}(\rho) \rightarrow +\infty$ for $\rho \rightarrow \infty$ and this we will assume.

Besides, by the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$|J_3| \leq J_1^{1/2} \left(\int_\rho^{+\infty} \frac{1}{2H(y)} \left(\int_{\Upsilon(y)} \partial_y^2 u^\perp(y, z) dz \right)^2 dy \right)^{1/2}.$$

We now deal with the inner integral in z in the last expression. To this end, we take into account the orthogonality conditions (19) and the trace inequality. We differentiate the first equality in (19) twice with respect to y and obtain

$$\sum_{\pm} \left(2\partial_y u^\perp(y, \pm H(y)) \partial_y H(y) + u^\perp(y, \pm H(y)) \partial_y^2 H(y) \pm \right.$$

$$\left. \pm \partial_z u^\perp(y, \pm H(y)) (\partial_y H(y))^2 \right) + \int_{\Upsilon(y)} \partial_y^2 u^\perp(y, z) dz = 0.$$

Thus,

$$\begin{aligned} \left(\int_{\Upsilon(y)} \partial_y^2 u^\perp(y, z) dz \right)^2 &\leq c \sum_{\pm} \left(|\partial_z u^\perp(y, \pm H(y))|^2 |\partial_y H(y)|^4 + \right. \\ &\quad \left. + |u^\perp(y, \pm H(y))|^2 |\partial_y^2 H(y)|^2 + |\partial_y u^\perp(y, \pm H(y))|^2 |\partial_y H(y)|^2 \right). \end{aligned}$$

For the first two terms between the brackets we use the trace inequality

$$\left| \partial_z u^\perp(y, \pm H(y)) \right|^2 |H(y)|^2 + \left| u^\perp(y, \pm H(y)) \right|^2 \leq c |H(y)|^3 \int_{\Upsilon(y)} \left| \partial_z^2 u^\perp(y, z) \right|^2 dz.$$

For the third term, we write down the chain of inequalities

$$\begin{aligned} \left| \partial_y u^\perp(y, \pm H(y)) \right|^2 &\leq c H(y) \int_{\Upsilon(y)} \left| \partial_y \partial_z u^\perp(y, z) \right|^2 dz + \\ &\quad + c |H(y)|^{-2} \left(\int_{\Upsilon(y)} \partial_y u^\perp(y, z) dz \right)^2 \leq c H(y) \int_{\Upsilon(y)} \left| \partial_y \partial_z u^\perp(y, z) \right|^2 dz + \\ &\quad + 2c \left(\frac{\partial_y H(y)}{H(y)} \right)^2 \left(\left| u^\perp(y, H(y)) \right|^2 + \left| u^\perp(y, -H(y)) \right|^2 \right). \end{aligned}$$

As a result, we find that

$$\begin{aligned} \left| \int_{\Upsilon(y)} \partial_y^2 u^\perp(y, z) dz \right|^2 &\leq c \left(|\partial_y H(y)|^4 H(y) + |\partial_y^2 H(y)|^2 |H(y)|^3 \right) \times \\ &\quad \times \int_{\Upsilon(y)} \left| \partial_z^2 u^\perp(y, z) \right|^2 dz + c |\partial_y H(y)|^2 |H(y)| \int_{\Upsilon(y)} \left| \partial_y \partial_z u^\perp(y, z) \right|^2 dz. \end{aligned}$$

The final inequality for the integral J_3 takes the form

$$|J_3| \leq c_1(\rho) J_1^{1/2} I_1^{1/2} + c_2(\rho) J_1^{1/2} K_1^{1/2} \quad (25)$$

where $K_1 = \left\| \partial_{yz}^2 u^\perp; L_2(\Pi_\rho) \right\|^2$ and

$$\begin{aligned} c_1(\rho) &= c \sup_{y \in [\rho, +\infty)} \left(|\partial_y H(y)|^2 + |\partial_y^2 H(y)| |H(y)| \right), \\ c_2(\rho) &= c \sup_{y \in [\rho, +\infty)} |\partial_y H(y)|. \end{aligned}$$

Note that both supremums tend to 0 for $\rho \rightarrow +\infty$. A similar argument shows that

$$|J_4| \leq c_1(\rho) J_2^{1/2} I_1^{1/2} + c_2(\rho) J_2^{1/2} K_1^{1/2}. \quad (26)$$

It remains to process the second term in (20), that is,

$$\begin{aligned} I_2 &= \int_{\Pi_\rho} \left| z \partial_y u_1(y) + \partial_y \partial_z u^\perp(y, z) \right|^2 dx = \\ &= \int_{\Pi_\rho} \left| \partial_y \partial_z u^\perp(y, z) \right|^2 dx + \int_{\Pi_\rho} |\partial_y u_1(y)|^2 dx + 2 \int_{\Pi_\rho} \partial_y u_1(y) \partial_y \partial_z u^\perp(y, z) dx. \\ &\quad \stackrel{\text{=:}}{=} K_1 \quad \quad \quad \stackrel{\text{=:}}{=} K_2 \quad \quad \quad \stackrel{\text{=:}}{=} 2K_3 \end{aligned}$$

So it follows that

$$K_1 = I_2 - K_2 - 2K_3 \leq I_2 + 2|K_3|. \quad (27)$$

We continue by estimating the integral K_3 :

$$\begin{aligned} |K_3| &= \left| \int_{\rho}^{+\infty} \partial_y u_1(y) \int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y, z) dz dy \right| \\ &\leq \left(\int_{\rho}^{+\infty} G_{H^3, \rho}(y) |\partial_y u_1(y)|^2 dy \right)^{\frac{1}{2}} \left(\int_{\rho}^{+\infty} G_{H^3, \rho}(y)^{-1} \left| \int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y, z) dz \right|^2 dy \right)^{\frac{1}{2}} \\ &\leq c J_2^{1/2} \left(\int_{\rho}^{+\infty} G_{H^3, \rho}(y)^{-1} \left| \int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y, z) dz \right|^2 dy \right)^{1/2}. \end{aligned}$$

Differentiating the second formula (19) with respect to y yields

$$\int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y, z) dz + \sum_{\pm} \partial_z u^\perp(y, \pm H(y)) \partial_y H(y) = 0.$$

By the trace inequality we find that

$$\begin{aligned} \left| \int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y, z) dz \right|^2 &= \left(\sum_{\pm} \partial_z u^\perp(y, \pm H(y)) \right)^2 |\partial_y H(y)|^2 \leq \\ &\leq c |H(y)| |\partial_y H(y)|^2 \int_{\Upsilon(y)} |\partial_z^2 u^\perp(y, z)|^2 dz. \end{aligned}$$

Thus, from the relation (17) which implies (16), we get

$$|K_3| \leq c \sup_{y \in [\rho, +\infty)} \left\{ |G_{H^3, \rho}(y)|^{-1/2} |\partial_y H(y)| |H(y)|^{1/2} \right\} J_2^{1/2} I_1^{1/2} \leq c J_2^{1/2} I_1^{1/2}. \quad (28)$$

We find by combining (27) and (28) that

$$K_1 \leq I_2 + c J_2^{1/2} I_1^{1/2}$$

and it holds with (25) respectively (26) that

$$|J_3| \leq c_1(\rho) J_1^{1/2} I_1^{1/2} + c_2(\rho) J_1^{1/2} \left(I_2 + c J_2^{1/2} I_1^{1/2} \right)^{1/2}, \quad (29)$$

$$|J_4| \leq c_1(\rho) J_2^{1/2} I_1^{1/2} + c_2(\rho) J_2^{1/2} \left(I_2 + c J_2^{1/2} I_1^{1/2} \right)^{1/2}. \quad (30)$$

Using first (20) and (22), next (29) and (30) for ρ large enough, and finally (21), (23) and (24) we conclude that indeed

$$\begin{aligned} \|\nabla_x^2 u; L^2(\Pi_\rho)\|^2 &= I_1 + 4I_2 + I_3 \geq \\ &\geq I_1 + 4I_2 + J_1 + J_2 + 2J_3 + 2J_4 \geq \\ &\geq \frac{1}{2} (I_1 + 4I_2 + J_1 + J_2) \geq \frac{1}{2} (I_1 + J_1 + J_2) \geq \\ &\geq c \min\{H^{-4}(\rho), W_H(\rho), W_{H^3}(\rho)\} \|u; L^2(\Pi_\rho)\|^2, \end{aligned}$$

whenever ρ is large enough. □

Theorem 8 Suppose that H satisfies Assumption 2 and that

$$\lim_{t \rightarrow \infty} \partial_t (\log (H(t))) = -\infty.$$

Then the embedding $H^2(\Omega)$ in $L^2(\Omega)$ is compact and the spectrum of the problem (4) with the Neumann boundary conditions (6) on both sides of the peak is discrete.

Now the following statement, which, as mentioned in the beginning of the paper, follows from a result in [1, 9].

Corollary 9 If (2) holds, then the spectrum of the problem (4), with the Neumann boundary conditions (6) on the sides of the peak, is discrete.

Remark 9.1 Note that the functions $H(y) = y^{-\alpha}$ and $H(y) = \exp(-\alpha y)$, $\alpha > 0$, do not satisfy the requirement in (2). The functions $H(y) = \exp(-y^{1+\alpha})$ with $\alpha > 0$ however do.

6 The incomplete Dirichlet condition and a 2d-estimate

Let the boundary conditions provide only the single stable condition

$$u(x) = 0, \quad x \in \Sigma_\rho^+ \quad (\text{or } x \in \Sigma_\rho^-). \quad (31)$$

Then Friedrich's inequality on section $\Upsilon(y)$ holds and, consequently,

$$\|\mathcal{A} \partial_z u; L^2(\Pi_\rho)\|^2 \geq c \|\mathcal{A} H^{-2} u; L^2(\Pi_\rho)\|^2$$

for every positive weight function $y \mapsto \mathcal{A}(y)$.

The function $v = \partial_z u$ can be represented as the sum $v(x) = v_0(y) + v^\perp(x)$ where, for $y > \rho$, the component v^\perp satisfies the first conditions in (19). Therefore,

$$\begin{aligned} \int_{\Pi_\rho} |\nabla_x^2 u(x)|^2 dx &\geq \int_{\Pi_\rho} |\nabla_x v(x)|^2 dx \geq \int_\rho^{+\infty} 2H(y) |\partial_y v_0(y)|^2 dy + \\ &+ \int_{\Pi_\rho} |\partial_z v^\perp(y, z)|^2 dx + 2 \int_{\Pi_\rho} \partial_y v_0(y) \partial_y v^\perp(y, z) dx =: I_4 + I_5 + 2I_6. \end{aligned}$$

Setting $Z_H(y) = H(y)^{-1} G_H(y)$, we get

$$I_4 \geq \int_\rho^{+\infty} 2G_H(y) |v_0(y)|^2 dy \geq \int_\rho^{+\infty} 2Z_H(y) H(y) |v_0(y)|^2 dy = \|Z_H v_0; L^2(\Pi_\rho)\|^2.$$

Friedrich's inequality implies $I_5 = \|\partial_z v^\perp; L_2(\Pi_\rho)\| \geq c \|H^{-2} v^\perp; L^2(\Pi_\rho)\|^2$. Furthermore,

$$\begin{aligned} I_6 &= \int_{\Pi_\rho} \partial_y v_0(y) \partial_y v^\perp(y, z) dx \leq I_5^{1/2} \left(\int_\rho^{+\infty} \frac{1}{2H(y)} \left| \int_{\Upsilon(y)} \partial_y v^\perp(y, z) dz \right|^2 dy \right)^{1/2} \\ &\leq I_5^{1/2} \left(\int_\rho^{+\infty} \frac{1}{2H(y)} |v^\perp(y, H(y)) \partial_y H(y) - v^\perp(y, -H(y)) \partial_y H(y)|^2 dy \right)^{1/2}. \end{aligned}$$

Thus $I_6 \leq c |\partial_y H(\rho)| I_5^{1/2} I_6^{1/2}$ holds and hence

$$\|\nabla_x^2 u; L^2(\Pi_\rho)\|^2 \geq c \|\min\{H^{-4}; H^{-3} G_H\} u; L^2(\Pi_\rho)\|^2.$$

Theorem 10 Suppose that $\lim_{y \rightarrow \infty} H(y)^{-3} G_H(y) = +\infty$. Then problem (4) with the boundary condition as in (31) has only a discrete spectrum.

Remark 10.1 The functions $H(y) = y^{-1-\alpha}$ with $\alpha > 0$ meet the condition in Theorem 10.

References

- [1] *R.A. Adams, J.J.F. Fournier*, Sobolev Spaces (Second Edition), Elsevier, 2009.
- [2] *F.L. Bakharev, S.A. Nazarov*, On the structure of the spectrum of a problem in the theory of elasticity for a body with a supersharp peak. (Russian. Russian summary) *Sibirsk. Mat. Zh.* 50 (2009), no. 4, 746–756; translation in *Sib. Math. J.* 50 (2009), no. 4, 587–595.
- [3] *M. van den Berg*, On the spectrum of the Dirichlet Laplacian for horn-shaped regions in \mathbb{R}^n with infinite volume, *J. Funct. Anal.* 58 (1984) 150–156.
- [4] *M. van den Berg, M. Lianantonakis*, Asymptotics for the spectrum of the Dirichlet Laplacian on horn-shaped regions. *Indiana Univ. Math. J.* 50 (2001), no. 1, 299–333.
- [5] *M.S. Birman, M.Z. Solomyak*, Spectral theory of selfadjoint operators in Hilbert space, Leningrad. Univ., Leningrad, 1980. (transl.: D. Reidel Publishing Co., Dordrecht, 1987.)
- [6] *S.I. Boyarchenko, S. Z. Levendorskii*, Spectral asymptotics of the Neumann Laplacian on oscillating horns. *Russ. J. Math. Phys.* 7 (2000), no. 1, 1–14.
- [7] *R. Courant and D. Hilbert*, Methods of Mathematical Physics, 1953.
- [8] *E.B. Davies, B. Simon*, Spectral properties of Neumann Laplacian of horns. *Geom. Funct. Anal.* 2 (1992), no. 1, 105–117.
- [9] *W.D. Evans, D.J. Harris*, Sobolev embeddings for generalized ridge domains, *Proc. London Math. Soc.* V. 3, 54, (1987) 141–175.
- [10] *F. Gazzola, H.-Ch. Grunau, G. Sweers*, Polyharmonic boundary value problems. Heidelberg, Dordrecht, London, New York: Springer, 2010.
- [11] *R. Hempel, L.A. Seco and B. Simon*, The essential spectrum of Neumann Laplacians on some bounded singular domains. *J. Funct. Anal.* 102 (1991), no. 2, 448–483.
- [12] *V. Ivrii*, Precise spectral asymptotics for Neumann Laplacian in domains with cusps, *Appl. Anal.* 71 (1999) 139–147.
- [13] *V. Jakšić, S. Molčanov and B. Simon*, Eigenvalue asymptotics of the Neumann Laplacian of regions and manifolds with cusps. *J. Funct. Anal.* 106 (1992), no. 1, 59–79.
- [14] *V. Jakšić*, On the spectrum of the Neumann Laplacian of long-range horns: a note on the Davies-Simon theorem. *Proc. Amer. Math. Soc.* 119 (1993), no. 2, 663–669.
- [15] *H. Kovarik*, Eigenvalue asymptotic of Robin Laplace operators on two-dimensional domains with cusps. *J. Lond. Math. Soc.* (2) 83 (2011), no. 1, 256–271.
- [16] *V.V. Krylov, F.J.B.S. Tilman*, Acoustic ‘black holes’ for flexural waves as effective vibration dampers, *Journal of Sound and Vibration* 274 (2004), 605–619.

- [17] *V.G. Maz'ya and S.V. Poborchii*, Imbedding and continuation theorems for functions in non-Lipschitz domains, S.-Petersburg. Gos. Univ., St. Petersburg, 2006.
- [18] *V.G. Maz'ya and S.V. Poborchii*, Differentiable functions on bad domains. World Scientific Publishing Co., Inc., River Edge, NJ, 1997.
- [19] *S.G. Mikhlin*, Variational methods in mathematical physics M.: Nauka, 1970.
- [20] *M.A. Mironov*, Propagation of a flexural wave in a plate whose thickness decreases smoothly to zero in a finite interval, Soviet Physics-Acoustics 34 (1988), 318–319.
- [21] *S.A. Nazarov*, Polynomial property of selfadjoint elliptic boundary value problems, and the algebraic description of their attributes. (Russian) Uspekhi Mat. Nauk 54 (1999), no. 5(329), 77–142; translation in Russian Math. Surveys 54 (1999), no. 5, 947–1014.
- [22] *S.A. Nazarov*, Asymptotic Theory of Thin Plates and Rods. Vol.1. Dimension Reduction and Integral Estimates. Nauchnaya Kniga: Novosibirsk. 2002.
- [23] *S.A. Nazarov*, On the spectrum of a problem in the theory of elasticity for a spiked body. (Russian) Sibirsk. Mat. Zh. 49 (2008), no. 5, 1105–1127; translation in Sib. Math. J. 49 (2008), no. 5, 874–893.
- [24] *S.A. Nazarov, J. Taskinen*, On the spectrum of the Steklov problem in a domain with a peak, Vestnik St.-Petersburg Univ. 2008. N 1 (Iss. 1). P. 56–65. (English transl.: Vestnik St.-Petersburg Univ. Math. 2008. V. 41, N 1. P. 45–52.)
- [25] *S.A. Nazarov*, On the essential spectrum of boundary value problems for systems of differential equations in a bounded domain with a cusp. (Russian) Funktsional. Anal. i Prilozhen. 43 (2009), no. 1, 55–67; translation in Funct. Anal. Appl. 43 (2009), no. 1, 44–54.
- [26] *F. Rellich*, Das Eigenwertproblem von $\Delta u + \lambda u = 0$ in Halbröhren, in Studies and Essays Presented to R. Courant, New York, pp. 329–344, 1948.
- [27] *G.V. Rozenbljum*, The eigenvalues of the first boundary value problem in unbounded domains. Mat. Sb. (N.S.) 89 (131) (1972), 234–247, 350.
- [28] *B. Simon*, The Neumann Laplacian of a jelly roll. Proc. A.M.S. 114, No. 3 (1992), 783–785.